The Dream of Captain Carib

By Jonathan Farley

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EMPEROR ABUBAKARI II GREETED ME AS I PASSED THE tropical gardens. "Oh Captain, my captain," he called, smiling. "I've discovered a truth-value that is neither true nor false, and yet both."

"With all due respect," I said, "you must be mistaken."

"Not at all. Here, let me show you." He took out a pad and pen and started scribbling. "You're familiar with the truth-functional connectives, 'and' (&), 'or' (V), 'if and only if' (≡)?"

p q	p&q	pVq	p≡q
T T	T	T	T
T F	F	T	F
F T	F	T	F
F F	F	F	T

1. "Now," Abubakari continued, "these are all associative. For instance,

$$(p\&q)\&r = p\&(q\&r),$$

since each side is true exactly when p,q, and r are all true.

"Is there another connective that is associative?" [Selected answers below.]

2. "That was easy," I said.

"Now, each of these connectives has an identity: for instance, if we let 'T' stand for 'true,' and 'F' for 'false,' then

$$pVF = p,$$

since both sides are true exactly when p is true. Similarly,

$$p\&T = p \text{ and } p\equiv T = p.$$

"For which connectives do you have inverses? That is, if p is an arbitrary truth-value, can we always find an inverse truth value to p - call it p^{-1} - such that

$$pVp^{-1} = F? \text{ Or } p\&p^{-1} = T? \text{ Or } p\equiv p^{-1} = T?"$$

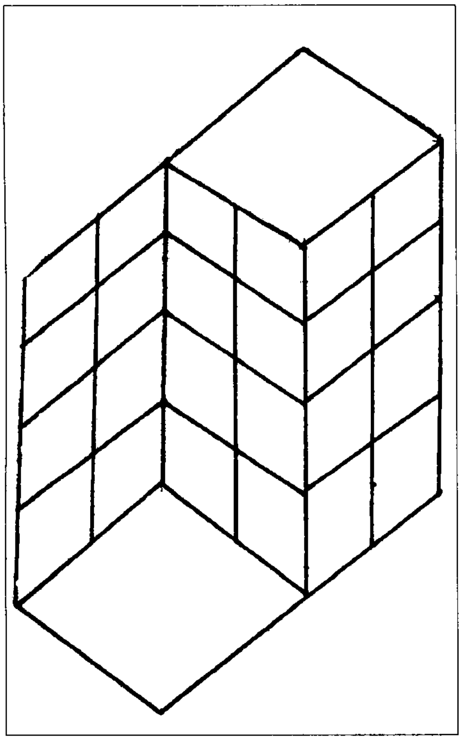
"What's your point?" I ventured to ask.

"My point is this: You proved '≡' and 'V' are associative, with identity elements T and F, respectively. In addition, a truth-value p has an inverse under '≡,' namely p itself. Thus the set {T,F} under '≡' forms a group."

"A what?"

"A group - like the set of integers under '+.' It is associative - (2+3)+4 = 2+(3+4) - it has an identity - 5+0 = 5 - and every integer has an inverse - 6+(-6) = 0."

"Oh," said I. "But you can multiply integers too. I suppose you're going to tell



me 'V' is like multiplication?"

"You're exactly right. Look at it this way:

+	0	1
0	0	1
1	1	0

•	0	1
0	0	1
1	1	0

"If '0' stands for 'even,' and '1' stands for 'odd,' this chart can be read as follows: when you add an even integer to an odd, you get an odd integer - i.e., 0 + 1 = 1 - when you multiply two even integers, you get an even integer, etc. Now compare these addition and multiplication tables with our truth tables for '≡' and 'V':"

≡	T	F
T	T	F
F	F	T

V	T	F
T	T	T
F	T	F

"Uncanny!" I exclaimed. "If you substitute '≡' for '+,' 'V' for '•,' 'T' for '0,' and 'F' for '1,' you get the exact same tables!"

"And just as multiplication is associative, has an identity '1,' and is distributive over addition, 'V' is associative, has an identity 'F,' and is distributive over '≡.'"

3. Prove that 'V' distributes over '≡': i.e., that

$$pV(q\equiv r) = (pVq)\equiv(pVr).$$

I said, "So our system of truth-values {0,1} - call it L for Logic - can do logical arithmetic with truth-values which mirrors numerical arithmetic with integers."

"In fact, our system can do something you can't do in the integers - division - because '1' is the only number you can divide by in L, and you can *always* divide by 1.

"So we can add and subtract, multiply and divide, almost as with real numbers. Mathematicians would call L a *field.*"

"Wow. But, er...what good is it?"

4. "Well," said his majesty," tell me: for what truth-values is (xVx)≡F true?"

"That was trivial with truth-tables," I told the emperor.

"Indeed, but here's a way to solve it using arithmetic. On the top line, we have the original problem. On the bottom, we convert '≡' to '+,' 'V' to '•,' 'T' to '0,' and 'F' to '1':

$$(xVx)\equiv F = T$$

$$(x\bullet x)+1 = 0$$

$$\text{i.e.:}\quad x^2 + 1 = 0.$$

This clearly has the solution x=1, as we can see if we recall that, in our special addition, 1+1=0 (the sum of two odd integers is even). Since '1' corresponds to 'F,' our solution is x=F."

"That was neat, but I thought '$x^2 + 1 = 0$' didn't have a solution."

"It doesn't," Abubakari clarified, "when 'x' is restricted to the real numbers. But here, 'x' is restricted to *truth values*, not real numbers at all. Obviously, '1' the number is not a solution of '$x^2 + 1 = 0$,' but '1' the truth-value is."

"So can we solve any polynomial using truth-values in L?"

5. "That's an interesting question. Can you find a polynomial with coefficients in L - that is, with coefficients that are 0 or 1 - that has no root? I.e., such that $f(x) \neq 0$ for all x in L?"

After a bit of scribbling on the monarch's pad, I found one.

"So what?"

Abubakari said, "Well, even though '$x^2 + 1$' has no root in the reals, it does have a root - namely $i = \sqrt{-1}$ - in an 'extension' of the reals, the complex number field. Similarly, we can imagine an extension of L in which '$x^2 + x + 1$' has a root."

I stared at him, perplexed.

"Don't you see?" he said. "'$x^2 + x + 1 = 0$' corresponds to '$(x \vee x) \equiv x \equiv F = T$.' This equation can never hold as long as 'x' is restricted to L, because whether you substitute 'T' of 'F' for 'x,' the left-hand side comes out false, whereas the right-hand side is, of course, true. But in our field extension of L - call it L^2 - '$x^2 + x + 1$' *does* have a root. This root - call it '*j*' - is such that

$$j^2 + j + 1 = 0,$$

that is,

$$(j \vee j) \equiv j \equiv F$$

is *true*!"

I stared at him perplexed.

"'*j*' is a new truth value!" he shouted at me. "Neither true nor false!"

"But what, then, is it, emperor? 'Maybe'? And what makes you think '*j*' corresponds to a truth-value? *Mathematically*, it may be a root of

$$'x^2 + x + 1,'$$

but, outside of that, it's totally meaningless."

"Like $\sqrt{-1}$?" he asked. "No, it may be without meaning, but it's not meaningless."

"So, do we have a three-valued logic?"

"No!" said Abubakari, grinning.

6. Find the other truth value. "Just as every complex number can be written as

$$a + bi$$

[or (a,b) for shorthand], where 'a' and 'b' are real numbers, so every truth-value in L^2 can be written as

$$a + bj$$

[or (a,b)], where 'a' and 'b' this time are truth-values in L, that is, either 0 or 1.

"Thus the truth-values in L^2 are:

$$0 + 0j = 0 \quad ('T')$$
$$1 + 0j = 1 \quad ('F')$$
$$0 + 1j = j \quad ('j')$$
$$1 + 1j = j + 1 \; ('j \equiv F')$$

I stared at him perplexed.

"'j' is a new truth value!" he shouted at me. "Neither true nor false!"

"But what, then, is it, emperor? 'Maybe'?

"So there are exactly four truth-values in all of time and earth!" I exclaimed.

Abubakari's eyes welled with tears. "Alas, nay. For any finite list of truth-values, there is a truth-value not on the list."

It seems 'j' is as different from '$j+1$,' the fourth truth-value, as truth is from falsehood."

"It is strange," I commented, "that to assert that

'j' is equivalent to 'F'

is to assert something that is neither true nor false nor even 'j,' whatever we are to take that to mean, since '$j+1$' is distinct from '0,' '1,' and 'j.'"

7. "You know," I said thoughtfully, "what polynomial corresponds to the fallacy 'x is true and x is false'?"

"Let's find out!" exclaimed the emperor. "To say 'p is false' is to say 'p is equivalent to F.' So when we want to negate a statement, we just attach '≡F' to it. So we read 'p≡F' as 'p is false.'

"Now, to assert 'p is true and q is false' is to assert that 'p is false or q is true' is *false*. Now, 'p is false or q is true' is clearly '(p≡F)Vq.' So '[(p≡F)Vq]≡F' says what we want, namely that

'p is false or q is true' is false.

If p and q happen to be the same, say, 'x,' then we have the statement '[(x≡F)Vx]≡F,' and hence the polynomial

$$[(x+1)\bullet x]+1 = x^2 + x + 1.$$

We already know this has the root 'j,' so 'j is true and j is false' is true!"

"In fact," I added, "since 'x' and 'x+1' have similar roles in your polynomial, then, since 'j' is a root, '$j+1$' must be also. Or, put another way, if we interpret 'x≡F' as 'x is false,' then 'j≡F' is 'j is false.' And so if 'j' is both true and false, the statement 'j is false' is both false and true, hence it too is a solution.

"It gets stranger," I continued. "Since 'j' and '$j+1$' simply negate each other, but are otherwise similar, like mirror images, we have no way of knowing which one of them is 'closer' in some sense to 'T,' and which is 'closer' to 'F' - which is the true form and which the reflection. Like identical twins, we know they're completely distinct, but we have no way to tell which one is which."

The emperor ignored me. "I'm curious about how our truth tables look when we include 'j' and '$j+1$.' The addition table is easy to write, since if (a,b) and (c,d) are two truth-values in L^2, with a,b,c,d each being either '0' or '1,' we add like so:

$$(a,b)+(c,d) = (a+c,b+d)$$

(where we remember, of course, that $1+1=0$)."

"To multiply, we need to know what 'j^2' is," I said.

The emperor answered, "Since

$$j^2 + j + 1 = 0,$$

it follows that

$$j^2 = j + 1."$$

"Don't you mean

$$j^2 = -j - 1?"$$

"I mean both," responded the emperor. "Since '$1+1=0$,' we have '$-1=1$.' Hence

$$-j - 1 = (-1)j + (-1) = (1)j + (1) = j + 1."$$

"Hm," I said. "That means that 'j or j' has the same truth value as 'j is false'; and to assert 'j or j' is quite different from asserting 'j' itself.

"Also," I went on, "since 'j' and '$j + 1$' are twins, if the second is the square of the first, then the first must be the square of the second:

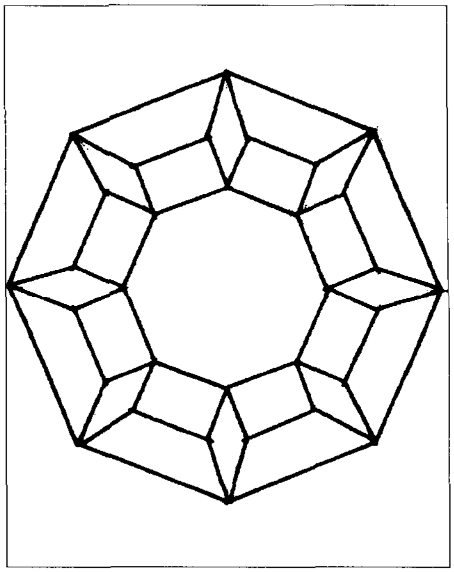
$$(j + 1)^2 = j$$

"All we need to know now, your majesty, is $j\,(j + 1)$ which is obvious, since

$$j^2 + j + 1 = 0 \text{ implies } j^2 + j = 1$$

and $j\,(j + 1) = j^2 + j$. Thus, the statement 'j is true or j is false' is false!

"Since 'T' is '0,' 'T' times any truth value is 'T,' and 'T' plus any truth value is that truth value. Similarly, since 'F' is '1,' 'F' times any truth value is that truth value."



≡ (+)	T	F	j	$j+1$
T	T	F	j	$j+1$
F	F	T	$j+1$	j
j	j	$j+1$	T	F
$j+1$	$j+1$	j	F	T

V (•)	T	F	j	$j+1$
T	T	T	T	T
F	T	F	j	$j+1$
j	T	j	$j+1$	F
$j+1$	T	$j+1$	F	j

"So there are exactly four truth-values in all of time and earth!" I exclaimed.

Abubakari's eyes welled with tears. "Alas, nay. For any finite list of truth-values, there is a truth-value not on the list."

8. Find a polynomial with no roots in L^2.

A tear slid off my cheek onto a wilted rose as the emperor explained.

"Given the list {T,F}, the statement

$$[(x \equiv T) V (x \equiv F)] \equiv F$$

is always false, if 'x' is restricted to the set {T,F}. It says, as you will recall, that the assertion 'x is true or x is false' is false; i.e., it says, 'x is not on the list.' We have already shown that this corresponds to the polynomial

$$[x \bullet (x+1)] + 1 = x^2 + x + 1.$$

"Similarly, given {T,F,j,$j + 1$}, the statement 'x is not on the list' is this:

$$[(x \equiv T) V (x \equiv F) V (x \equiv j) V (x \equiv j + 1)] \equiv F$$

This corresponds to the polynomial

$$x^4 + x + 1,$$

which therefore has no root in {T,F,j,$j + 1$}. Since we can do this with any finite set, there must be an infinite number of truth-values."

9. **(The Unproblem)** "What if we constructed a statement that could not be true in any field extension of L^2?" I asked. "For instance: 'x is the multiplicative inverse of j, but x is not $j + 1$.'"

"Why can that never be true?"

"Because we know that '$j\,(j + 1) = 1$,' and if '$xj = 1$' were true also, then

$$x = x \bullet 1 = x \bullet [j\,(j + 1)] = (xj) \bullet (j + 1) = 1 \bullet (j + 1) = j + 1$$

"As before, if we let 'p' be 'x is the multiplicative inverse of j,' and let 'q' be 'x is $j + 1$,' then the statement we want - 'p is true and q is false' - is '[(p≡F)Vq]≡F,' or '[(p+1)•q]+1.' Now, in symbols, 'p' is '$xj = 1$,' or 'xj is equivalent to 1.' Since equivalence is '+,' 'p' is actually '$xj + 1$.' And 'q' is

Selected Answers

1. 'p if and only if not q'
2. $p \equiv p = T$
4. $x = F$
5. $x^2 + x + 1$
6. $j \equiv F$
7. $x^2 + x + 1$
8. $x^2 + x + 1$

'$x \equiv (j + 1)$,' or '$x + (j + 1)$'. In sum, our statement is

$$[(xj + 1 + 1) \bullet (x + j + 1)] + 1 = j x^2 + x + 1$$

"But this polynomial, like any polynomial, *must* have a root in *some* extension of L^2! The way I see it, our reading '$p \equiv F$' as 'p is false' must be wrong. This paradox of negation is perplexing!"

10. **(Q-Continuum)** "Another thing," I continued. "The set of all truth values is infinite; but is it countably infinite like the integers and rational numbers (Q), or uncountably infinite like the continuum of real numbers? If the latter, then there are as many truth-values between 'T' and 'F' as there are real numbers between 0 and 1!"

11. **(Carib's Conjecture)** "I claim," said I, "that for any finite field of truth-values, there is a *quadratic* polynomial which has no roots in that field. (Since quadratics have two roots, this suggests every truth-value comes with a natural counterpart; might this conjugate be its negation?)

"Thus, it seems that a three-valued logic - or any logic with a finite number of truth-values that is not a power of 2 - is 'illogical.'"

THE EMPEROR FROWNED AT ME. "YOU'RE CLEVER, much cleverer than Lao or Werner." He turned and walked away stiffly. Then he stopped and turned around to face me, smiling. "I think I'll have you executed this afternoon. Good day, Captain Carib." φ