What is it that pure mathematicians do? Ask biologists what they study and they can point to living creatures in the world around us or perhaps take you to their lab to look through a microscope. Ask physicists, and they will talk about the many laws explaining everyday physical phenomena. Applied mathematicians can demonstrate their computer models of real-world occurrences or prove to you the actual utility of calculus. Ask ten pure mathematicians what they do, and you will probably get twenty different answers. Some might reference the study of patterns, others perhaps the analysis of abstract structure and symmetry. They will agree that proofs are central to what they do, but disagree on what constitutes a proof. They will insist that “real” mathematics has very little to do with the repetition of multiplication tables and formula memorization, familiar topics from grade school math classes, perhaps adding that mathematics, to them, is a fun game. No amount of philosophizing or semantic debate will help you to understand what mathematics really is—you have to see math being done and draw your own conclusions.

Here are a few examples of things some mathematicians have found interesting to help you on your way.

**Graph Theory: A Case Study in Structure**

Consider Figure 1. Each shape in the picture is a drawing of what mathematicians call a graph: a set of points, called vertices, some of which may be connected by an edge. The two drawings you see actually represent the same graph because, to a mathematician who studies these structures, the only relevant information is the set of vertices and which ones are connected. For a mathematician, barring a few who study graphs with specified edge lengths, it does not matter whether the edge connecting two vertices is curvy or straight, or how long it is—only that the vertices are connected.

A natural question to ask about two drawings of a graph would be: how can we tell when two drawings represent the same graph? In the case of figure 1, we could do discern this by sight alone, but often the drawings are more complicated. Similarly, if we were to number the vertices of each graph...
drawing, we might think they were different graphs because we numbered them differently, but since they have the same connections and vertices, they are the same graph. All of this relates to the problem of finding when two mathematical objects are isomorphic, essentially meaning that they have the same mathematical structure and properties, maybe with the names of the elements reordered. One of the open problems in graph theory is to figure out how many non-isomorphic graphs there are with n vertices. For every pair of vertices, we have exactly two choices, either to put an edge between them or not. For a small n, we can just list out all possible graphs and determine which ones are really the same—but this brute-force method quickly becomes unfeasible as n gets bigger. It is also mathematically unsatisfying, since the method of exhausting all possibilities can never give us the answer for an arbitrary choice of n.

Other questions we might ask about graphs involve how we can draw them. For example, the graph drawing in Figure 2 has some edges crossing. Is it possible to draw this graph in the plane (on a flat piece of paper) without any edges crossing? Try it, and you will likely think it impossible—mathematical proof can show conclusively that it is impossible. Now we have finally seen something useful come out of pure math—knowing that this graph can never be drawn without edges crossing saves computer scientists the trouble of trying to devise an algorithm to create such a drawing. It is also a prime example of another ubiquitous concept in mathematics: invariants. Given two representations of some structure, we can figure out that they represent different objects if we know that some invariant property of those structures is different for each representation. In this example, we know that any graph drawn in the plane without any edges crossing is not this graph.

As a more concrete example of a problem in graph theory, consider a party with six people. We will show that there exist some three people who are all mutual strangers, or some three people who are all mutual friends. We can do this by considering a graph with six vertices and all possible edges, and coloring the edges of the graph.

Draw all 15 possible edges between each of the six vertices in our graph. Color an edge red if the two vertices they connect represent mutual friends and blue if they are mutual strangers. Consider one particular vertex, which has five edges coming out of it, each colored red or blue. (See figure 3) There must be at least three edges coming out of that vertex that are all the same color, say red, since otherwise there would be at most two edges of each color, resulting in four total edges—but we know for sure that we have five edges. Let us imagine that the there are three red edges coming out of one vertex that connect to three different vertices. If any of these three vertices are connected by another red edge, then our graph has a red triangle, and thus has three mutual friends. If none of the edges are red, then they are all blue, and thus our graph has a blue triangle representing three mutual strangers.

After making this conclusion, the next step for a mathematician might be to consider whether the same thing works if we only have five people, or how many mutual friends there must be in a larger graph. Relating to graphs,
mathematicians can ask questions about invariants, the kind of substructures a graph is forced to have if it is big enough, the properties of graphs whose vertices or edges are colored in a certain configuration (the specific colors are irrelevant, but their arrangement matters), or what happens when we try to draw graphs on donuts or other strange surfaces.

The proofs we just did are not the kind of formulaic proofs most people see in high school geometry. At one time, mathematicians such as David Hilbert were very concerned with putting all mathematical knowledge in a completely rigorous framework where everything could be proven from a set of axioms. Once the logician Kurt Godel suggested his Incompleteness Theorem in 1931, showing that for any sufficiently powerful logical system there exist true statements that cannot be proven within the system, the so-called “formalists” had to abandon their goal.

Our proofs are the kind of proofs very common in modern mathematics, where many consider a proof simply a convincing argument that some mathematical statement is true. At the same time, mathematicians want to make sure that their proofs are correct and that there are not unusual examples of objects which make their conclusions false so they are careful to formulate their statements precisely. Mathematical proofs are often initially dreamt up as informal arguments, but they must eventually be made rigorous, especially when the results are counterintuitive. This balance between rigor and insight in proofs is one of the most important concerns for most mathematicians when they write proofs—the best mathematical arguments are not only technically correct, but also convincing and insightful.

Different Sizes of Infinity

What do we mean when we talk about infinity? Many people have an intuitive idea that infinity is what you get to when numbers get really big—despite being told that infinity is not a number, they think of it as the biggest number. But what do we mean when we talk about infinity, really? Are there different kinds of infinity?

Think about the natural numbers—0, 1, 2, etc. Contained in that “et cetera” is our intuitive idea of what it means to go on infinitely—we think the natural numbers just start at zero and keep going. Now let us consider the even natural numbers, 0, 2, 4, etc. Both of these sets have an “infinite” size when we just go by intuition, but infinity is a wishy-washy concept for many people—are we really talking about the same “infinities” in this case? We do know we can take an element of the natural numbers and associate it with exactly one even number, and given an even number we can obtain a unique natural number by dividing it by two. This means we have an isomorphism between the even
We have now shown that there are, in fact, different sizes of infinity, and our insistence on formalizing the idea of infinity was not a trivial concern.

**Numbers**

The study of the properties of numbers is one of the oldest areas of mathematics, and also the one that people think they are most familiar with. After all, isn’t mastering long-division the culmination of the study of numbers?

When number theorists study numbers, they study the structure of numbers and number patterns, often working with whole numbers. A central area of research in modern mathematics involves the study of prime numbers—numbers that cannot be divided evenly by anything except 1 and themselves—and how frequently they appear among the natural numbers. Are there infinitely many of them?

Suppose there are a finite quantity of prime numbers. Let us make a list of all primes and let p be the biggest number in our list. Now take the product of all the primes on the list and add one. If that new number is prime, then we have found a prime that is larger than any element of our supposedly-exhaustive list. Because we can repeat this process with any finite list of primes, we have shown that there are infinitely many primes. If the new number is not prime, we still know that it is not evenly divisible by any of the primes in our finite list, since we took their product and added one, and no integer greater than one divides evenly into 1. Since every number can be factored into a product of primes, we then know that there is a prime larger than any of the primes in our list. Now we know there are infinitely many primes.

To take an elementary example of the distribution of primes, let us consider a birthday problem. A man has three children who will be ages 3, 5, and 7 next year. Is there any other time when all of their ages will be prime again? Consider a number n greater than 3. If n is divisible by 3, then it is not prime. If it is one more than a multiple of three, then n+2 is a multiple of three, and thus not prime. If it is two more than a multiple of three, then n+4 is divisible by three. Thus, there is no set of three naturals two integers apart all of which are prime except for 3, 5, and 7.

Although the study of primes has recently become popular in cryptography, mathematicians began studying primes simply because they needed to know what kind of structure the primes have—for their curiosity, and for the fun of the game of discovering mathematical truth.

**What Next?**

Hopefully you have seen a little to peak your interest in the kinds of questions pure mathematicians study. Perhaps the most important thing to understand is that mathematics is not simply a body of facts or methods, but a way of making the ubiquitous concepts of number, pattern, and symmetry more precise, as well as a fun way of looking at the world. Mathematics itself is a creative process that is open to anyone willing to ask questions and seek answers.

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**References**